WEEKLY PROBLEM DECEMBER 6 TO DECEMBER 12 2009

DIVISION WITH REMAINDER AND BINOMIAL COEFFICIENTS

At some point in our high school careers we learn how to multiply expressions of the form

$$(a+b)^2$$

Right away we are told **NOT** to make the mistake of saying that

$$(a+b)^2 = a^2 + b^2$$

Instead we are told to use the FOIL method which says that

$$(a+b)^2 = (a+b) \times (a+b) = a^2 + 2ab + b^2$$

But what if we want to find out what $(a+b)^3$ is? Or more generally what if we want to find

$$(a+b)^n = \underbrace{(a+b) \times (a+b) \times \cdots \times (a+b)}_{n \text{ copies}}$$

We can compute again using the FOIL method to get the following.

$$\begin{aligned}
(a+b)^0 &= \mathbf{1} \\
(a+b)^1 &= \mathbf{1} \cdot a + \mathbf{1} \cdot b \\
(a+b)^2 &= \mathbf{1} \cdot a^2 + \mathbf{2} \cdot ab + \mathbf{1} \cdot b^2 \\
(a+b)^3 &= \mathbf{1} \cdot a^3 + \mathbf{3} \cdot a^2 b + \mathbf{3} \cdot ab^2 + \mathbf{1} \cdot b^3 \\
(a+b)^4 &= \mathbf{1} \cdot a^4 + \mathbf{4} \cdot a^3 b + \mathbf{6} \cdot a^2 b^2 + \mathbf{4} \cdot ab^3 + \mathbf{1} \cdot b^4 \\
(a+b)^5 &= \mathbf{1} \cdot a^5 + \mathbf{5} \cdot a^4 b + \mathbf{10} \cdot a^3 b^2 + \mathbf{10} \cdot a^2 b^3 + \mathbf{5} \cdot ab^4 + \mathbf{1} \cdot b^5 \\
(a+b)^6 &= \mathbf{1} \cdot a^6 + \mathbf{6} \cdot a^5 b + \mathbf{15} \cdot a^4 b^2 + \mathbf{20} \cdot a^3 b^3 + \mathbf{15} \cdot a^2 b^4 + \mathbf{6} \cdot ab^5 + \mathbf{1} \cdot b^6 \\
(a+b)^7 &= \mathbf{1} \cdot a^7 + \mathbf{7} \cdot a^6 b + \mathbf{21} \cdot a^5 b^2 + \mathbf{35} \cdot a^4 b^3 + \mathbf{35} \cdot a^3 b^4 + \mathbf{21} \cdot a^2 b^5 + \mathbf{7} \cdot ab^6 + \mathbf{1} \cdot b^7
\end{aligned}$$

Do you notice the pattern that occurs with the boldface coefficients? These coefficients are aptly named *binomial coefficients* and have a relation to probability in the following way. For n, k integers [1], we define

$$\begin{aligned}
\binom{n}{k} &\overset{\text{def}}{=} \text{The number of ways to choose } k \text{ objects from } n \text{ objects} \\
&= \frac{n!}{k! \cdot (n-k)!} \qquad \text{(where ! is the factorial function)} \\
&= \frac{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot k}
\end{aligned}$$

Please show that

$$(a+b)^n = a^n + \binom{n}{1} \cdot a^{n-1} b + \binom{n}{2} \cdot a^{n-2} b^2 + \cdots + \binom{n}{k} \cdot a^{n-k} b^k + \cdots + \binom{n}{n-2} \cdot a^2 b^{n-2} + \binom{n}{n-1} \cdot ab^{n-1} + b^n$$

Now suppose that a, b are integers, and consider the remainder of $(a+b)^n$ when this expression is divided by n. For which integers n are the remainders (after dividing by n) of $(a+b)^n$ and $(a^n + b^n)$ the same? (HINT: It has to do with how n divides the binomial coefficients). Notice that for such n it **IS** true that

$$(a+b)^n = a^n + b^n$$

in some sense.

[1] There is also a generalization of binomial coefficients where n and k are complex numbers.